Financial Econometrics
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Arithmetic vs logarithmic returns

Preliminaries

A **rate of return**, **return on investment**, **rate of profit**, or simply **return**, is a measure of how much was gained over a period of time. Arithmetic returns represent the "intuitive" way in which we think about gains and losses. For example, if we buy one share of company XYZ for \$100 today and sell it in a year from now, the (yearly) **arithmetic** return on that investment R_{t+1} is given by

$$R_{t+1} = \frac{P_{t+1} - P_t}{P_t} = \frac{110 - 100}{100} = 0.1 = 10\%,$$

where P_t represents the price at time t, while the t and $t+1$ subscripts represent the dates today and in the next period (in this case, one year).

To understand the difference between arithmetic and logarithmic returns, we need to introduce the notion of **compounding**. To understand this notion, we assume a fixed yearly arithmetic return of 10% per year, with an original investment value $P_t = \$100$.

- With **annual compounding**, the interest is paid *at the end of the period*, so at the end of the year the investor gets an amount equal to $\$100(1.1) = \$110$.
- With **semiannual compounding**, the investor is paid 5% *every six months*. So the original \$100 grows to $\$100(1.05)(1.05) = \$110.25$ at the end of the year. Note that due to the compounding, the \$5 gained during the first six months will earn interest during the following six months.
- With **quarterly compounding**, i.e., payments every quarter (3 months), the investor now gets $\$100(1.025)^4 = \$110.38$.

More generally, if we have an amount P_t invested for one year at a rate R per year, and this rate is compounded m times per year, the final payment P_{t+1} at the end of the year is

$$P_{t+1} = P_t\Big(1 + \frac{R}{m}\Big)^m$$

With $P_t = 100$, $R = 0.1 = 10\%$, and $m = 1, 2, 4$, we get the examples above. Now, if we compound the interest at ever decreasing time intervals, that is, let $m \to \infty$, then elementary calculus tells us that P_{t+1} becomes

$$P_{t+1} = P_t e^R$$

Where e is the base of the natural logarithm. In this case, the rate R is called a **continuously compounded rate of return**. Note that

$$\ln\Big(\frac{P_{t+1}}{P_t}\Big) = \ln\Big(\frac{P_t e^R}{P_t}\Big) = R$$

Since P_{t+1} stands for the value of the investment at the end of the investment period and P_t stands for the value of the investment at the beginning of the investment period, we define the logarithmic return r_{t+1} at time $t+1$ as

$$r_{t+1} = \ln\left(\frac{P_{t+1}}{P_t}\right)$$

So, in essence, the logarithmic return provides the corresponding continuously compounded return of the investment, while the arithmetic return simply provides the return without any compounding during the investment period, just like the annual investment in the example above. Nevertheless, it is important to notice that arithmetic and logarithmic returns tend to be fairly similar for relatively small R.

Multiple-period returns

Of course, the discussion above can be extended to multiple periods, i.e., multiple years, weeks, months, or whatever basic time unit you consider. The following list summarizes the results:

- As before, P_t is the price at date t
- One-period arithmetic return: $R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$
- One-period logarithmic return: $r_t = \ln\left(\frac{P_t}{P_{t-1}}\right)$
- k-period arithmetic return: $R_t[k] = \frac{P_t - P_{t-k}}{P_{t-k}}$
- k-period logarithmic return: $r_t[k] = \ln\left(\frac{P_t}{P_{t-k}}\right)$
- An important advantage of logarithmic returns is that the multiple-period return is simply the sum of one-period returns: $r_t[k] = \sum_{j=0}^{k-1} r_{t-j}$.

The last property implies that working with log-returns is sometimes preferable in the context of time series analysis, since one-period log-returns can easily be aggregated into multiple period returns.

Portfolio returns

On the other hand, when we consider portfolios of stocks, it is often convenient to use arithmetic returns instead of log-returns. Reasons?

- (Check this!) The arithmetic return of an N-asset portfolio is the weighted average of the **arithmetic** returns of the N assets. Assume $\sum_{i=1}^{N} \omega_i = 1$ and let p be the portfolio with weight ω_i on asset i. Then,

  $$R_{p,t} = \sum_{i=1}^{N} \omega_i R_{i,t}.$$

- (Check this!) Logarithmic returns **do not** have the same convenient property, i.e.,

  $$r_{p,t} \neq \sum_{i=1}^{N} \omega_i r_{i,t}.$$